# Integer Programming Formulation of the Problem of Generating Milton Babbitt's All-partition Arrays

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## 1. Objective

- Our aim is to generate the 12-tone compositional structure developed by Milton Babbitt known as an ***all-partition array***.
- An all-partition array is a covering of an $I \times J$ pitch-class matrix by a collection of subsets, each containing an ***aggregate*** and each represented by a distinct ***integer partition*** of 12.

## 2. Our method

- We formulate the problem of generating an all-partition array using ***integer programming (IP)***, a powerful programming paradigm in which problems are described with discrete linear equations and/or inequalities.
- We introduce the use of *overlaps* between subsets, instead of *insertions*, which allow us to define the generation of an all-partition array as a ***set-covering problem (SCP)***.

## 3. Introducing Overlaps

- An all-partition array's matrix contains fewer elements than does its required collection of subsets. Therefore, additional pitch classes must be found.
- Original construction:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 4 | 3 | 3 | 5 | 9 | 10 | 1 | 8 | 2 | 0 | 7 | 6 | | | |
| 6 | 7 | 7 | 0 | 2 | 8 | 1 | 10 | 9 | 5 | 3 | 4 | 11 | | | |
| 5 | 6 | 11 | 1 | 7 | 0 | 9 | 8 | 4 | 2 | 3 | 10 | | | | |
| 2 | 9 | 10 | 10 | 8 | 4 | 3 | 0 | 5 | 11 | 1 | 6 | 7 | | | |
| 0 | 5 | 4 | 6 | 10 | 11 | 2 | 9 | 3 | 1 | 8 | 7 | | | | |
| 1 | 8 | 9 | 7 | 3 | 2 | 11 | 4 | 10 | 0 | 5 | 6 | | | | |

**Figure 1:** Subsets represented by the integer partitions $[3,3,2,2,1,1]$ and $[3,3,3,3]$. Insertions of additional pitch classes make the matrix irregular.

**V.S.**

- Our formulation:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 4 | 3 | 5 | 9 | 10 | 1 | 8 | 2 | 0 | 7 | 6 |
| 6 | 7 | 0 | 2 | 8 | 1 | 10 | 9 | 5 | 3 | 4 | 11 |
| 5 | 6 | 11 | 1 | 7 | 0 | 9 | 8 | 4 | 2 | 3 | 10 |
| 2 | 9 | 10 | 8 | 4 | 3 | 0 | 5 | 11 | 1 | 6 | 7 |
| 0 | 5 | 4 | 6 | 10 | 11 | 2 | 9 | 3 | 1 | 8 | 7 |
| 1 | 8 | 9 | 7 | 3 | 2 | 11 | 4 | 10 | 0 | 5 | 6 |

**Figure 2:** These same subsets with overlaps allowing for the matrix to remain regular.

## 4. Subsets $C_k$ & Constraints

- A binary variable $x_{i,j,k}$ indicates whether or not $(i,j)$ in the matrix belongs to a $k$th subset, $C_k$.
- $C_k$ is then the set of all $(i,j)$ where $x_{i,j,k} = 1$.

| | | | |
|---|---|---|---|
| 11 | 4 | 3 | 5 |
| 6 | 7 | 0 | 2 |
| 5 | 6 | 11 | 1 |
| 2 | 9 | 10 | 8 |
| 0 | 5 | 4 | 6 |
| 1 | 8 | 9 | 7 |

**(a)** $k$th subset.

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 |

**(b)** All $x_{i,j,k} = 1$ for $k$th subset.

| | | |
|---|---|---|
| (1,1) | (2,2) | (3,3) |
| (2,1) | (2,2) | |
| (3,1) | | |
| (4,1) | (4,2) | (4,3) |
| (5,1) | | |
| (6,1) | (6,2) | |

**(c)** $C_k$.

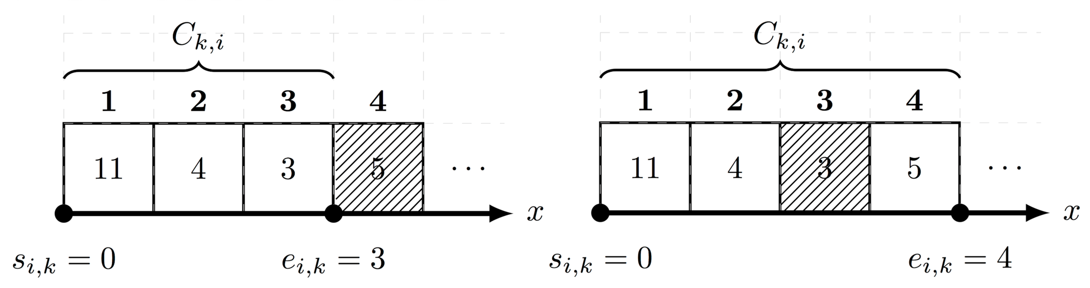
**Figure 3**

### Solution Constraints

1. Consecutiveness of row-elements in $C_k$.
2. Containment of 12 pitch classes in $C_k$.
3. Covering of all matrix elements by 58 $C_k$.
4. Restrictions on overlaps in contiguous $C_k$.
5. Distinctness of $C_k$ integer partitions.

## 5. IP Formulation

1. **Consecutiveness**




The elements of $C_{k,i}$ are found in a range from a starting point $s_k$ to ending point $e_k$:

$$\forall i \in [1,I], \forall k \in [1,K],\ 0 \le s_{i,k} \le e_{i,k} \le J.$$

Any element of $C_{k,i}$ is located to the lefthand side of $e_{i,k}$:

$$\forall i \in [1,I], \forall j \in [1,J], \forall k \in [1,K],\ \ j \cdot x_{i,j,k} \le e_{i,k},$$

Any element of $C_{k,i}$ is located to the righthand side of $s_{i,k}$:

$$\forall i \in [1,I], \forall j \in [1,J], \forall k \in [1,K],\ \ J - s_{i,k} \ge (J+1-j)\cdot x_{i,j,k},$$

The length of $C_{k,i}$ is exactly $e_{i,k} - s_{i,k}$:

$$\forall i \in [1,I], \forall k \in [1,K],\ \sum_{j=1}^{J} x_{i,j,k} = e_{i,k} - s_{i,k}.$$

## 5. IP Formulation (cont.)

2. **Containment**

Exactly 12 distinct pitch classes are contained in $C_k$:

$$\forall p \in [0,11], \forall k \in [1,K],\ \sum_{i=1}^{I}\sum_{j=1}^{J} B^{p}_{i,j} \cdot x_{i,j,k} = 1,$$

where $B^P$ contains all $(i,j)$ in the matrix corresponding to pitch class, $p$.

3. **Covering**

Each element in the matrix is covered at least once:

$$\forall i \in [1,I], \forall j \in [1,J],\ \sum_{k=1}^{K} x_{i,j,k} \ge 1.$$

4. **Restrictions on overlaps**

Contiguous $C_k$ subsets can not overlap by more than one location in $C_{k,i}$:

$$\forall i \in [1,I], \forall k \in [2,K],\ e_{i,k-1} \le e_{i,k},$$

$$\forall i \in [1,I], \forall k \in [2,K],\ e_{i,k-1} - 1 \le s_{i,k} \le e_{i,k-1},$$

5. **Distinctness**

Variable $y_{i,k,l}$ allows us to convert the horizontal lengths of $C_{k,i}$ into column sums, where:

$$\forall i \in [1,I], \forall k \in [1,K],\ e_{i,k} - s_{i,k} = \sum_{l=1}^{L} y_{i,k,l},$$

$$\forall i \in [1,I], \forall k \in [1,K], \forall l \in [2,L],\ \ y_{i,k,l-1} \ge y_{i,k,l},$$

such that $\forall i \in [1,I],\ y_{i,k,l}$ becomes Figure 3(b) and $\sum_{i=1}^{I} y_{i,k,l} = [6,4,2,0,0,0,0,0,0,0,0,0]$. $P_{n,l}$ $(1 \le l \le L)$ specifies the shape of $C_k$ as these column sums. Binary variable $z_{k,n}$ indicates whether $C_k$ takes the shape $P_{n,l}$ or not:

$$\forall k \in [1,K], \forall n \in [1,N], \forall l \in [1,L],\ \ P_{n,l} \cdot z_{k,n} \le \sum_{i=1}^{I} y_{i,k,l}.$$

All $C_k$ correspond to a distinct integer partition:

$$\forall n \in [1,N],\ \sum_{k=1}^{K} z_{k,n} = 1.$$

## 8. Solving smaller problems

Solving for the entire problem (i.e., $6 \times 96$ matrix and 58 subsets), proved too difficult. Therefore, we solved for smaller problems consisting of the first $J$ columns of the original matrix, a number of subsets equal to $(J+2)/2$, and 12 overlaps.

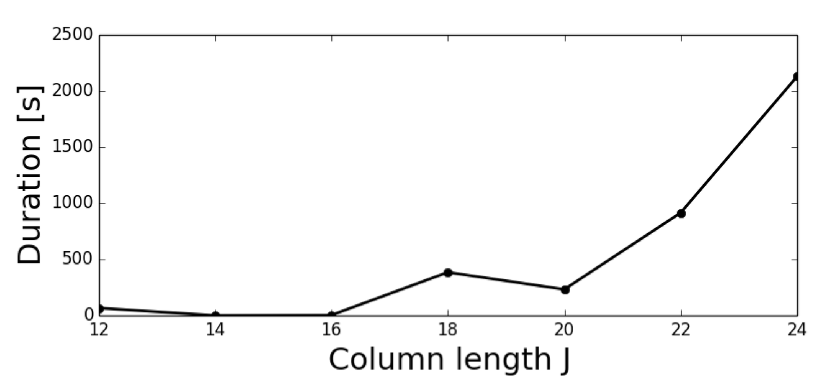
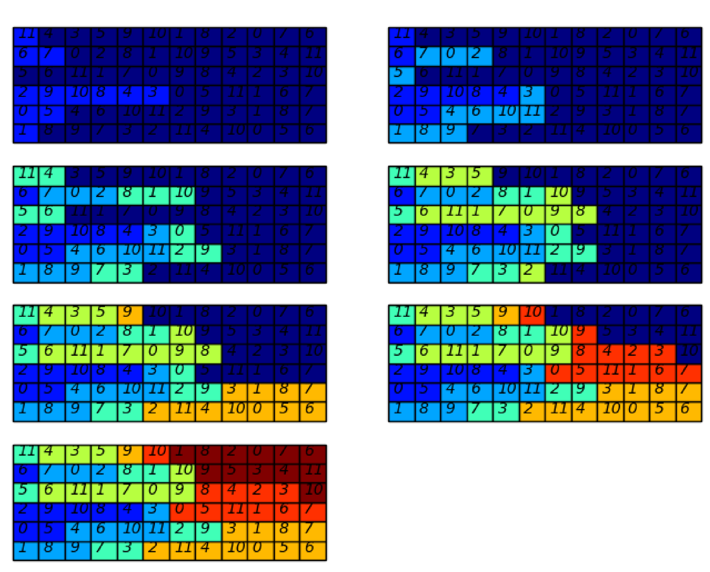



**Figure 4:** Solving time for smaller problems.

The figure below shows one solution for a smaller matrix ($J = 12$, 7 subsets). In future work, we hope to combine such smaller solutions to form a complete solution to the entire problem.



## 9. Acknowledgements/Refs

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